An Extension of Lauwerier’s Solution for Heat Flow in Saturated Porous Media

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Abstract: One of the crucial topics in this century is sustainable energy. Since the sources of fossil fuels are limited and are going to be exhausted, there is a need to look for sustainable renewable energy. In this respect, the exploitation of geothermal energy from deep hot aquifers becomes opportune. Hence, insight is required in the heat balance of potential aquifer systems. Essential issues are convection, conduction and dispersion.

Modeling such processes is affected by numerical errors when using computer models and by the complexity of analytical solutions. This article focuses on Lauwerier’s problem. As an extension, it is suggested that beside convection in the aquifer and conduction to adjacent layers also conduction in an aquifer can be considered in a simple way. For a characteristic situation, a comparison is made with the result of the numerical code COMSOL. This gives new insight in the possible misjudges of heat transport simulations due to numerical effects and in the applicability of models.

Keywords: heat transfer, porous media, analytical solution, heat loss (bleeding).

1. Introduction

Geothermal energy is a promising area for substitution of fossil fuels. Recently, much effort is given to optimize production by various modeling techniques. Analytical models are useful for obtaining a general view of principle effects and numerical calulations are proper techniques for the evaluation of complex situations. However, analytical models have serious limitations considering realistic applications and numerical models hamper from numerical errors and instability. For proper understanding of these errors, and for validating the numerical models, it is useful to regularly compare analytical and numerical results.

An analytical solution for convective heat transport in porous media is given by Lauwerier (1955). An extension to Lauwerier and a complete solution for convective-conductive heat flow together with bleeding to adjacent layers has recently been developed by Barends (2009). These analytical solutions have been used to identify the numerical errors.

2. Mathematical formulation

Figure 1. Schematization of the heat process in a two-layer system

The quasi two-dimensional convection-conduction heat balance equation in a plane-symmetric one-dimensional aquifer bounded by a conducting impervious adjacent layer (aquiclude) is described by following set of equations, assuming instant thermal equilibrium between fluid and grains.

\[
\begin{align*}
D \frac{\partial^2 T}{\partial x^2} - v \frac{\partial T}{\partial x} + \frac{Q}{H} = \frac{\partial T}{\partial t} + \frac{h}{H} q_0 & \quad (1a) \\
D \frac{\partial^2 T}{\partial z^2} = \frac{h}{\partial t} & \quad (1b) \\
\text{Bleeding: } h q_0 = D \frac{\partial T}{\partial z} & \quad (1c)
\end{align*}
\]

where \( x > 0, t > 0 \)

where \( z > 0, t > 0 \)

at the interface \( z = 0, t > 0 \)

Here:
\[
D = \frac{\lambda}{\rho c}, D' = \frac{\lambda'}{\rho c}, h = (\rho c)',
\]
\[
(\rho c)' = \left(\frac{\phi_f f_c f + (1 - \phi) \rho_s c_s}{\rho c}\right)
\]
\[
v = Rw, \quad w = \frac{q}{\phi}, \quad R = \frac{\phi f_c f}{\rho c}
\]

Where:
- \(T\) temperature [°C]
- \(Q\) heat source (heat production per meter width)
- \(H\) aquifer height [m]
- \(R\) thermal retardation factor
- \(T_i\) injected temperature [°C]
- \(T_0\) initial temperature [°C]
- \(\lambda\) heat conductivity [J/(ms°C)]
- \(\rho\) density [kg/m³]
- \(c\) specific heat capacity [J/(kg°C)]
- \(w\) real velocity [m/s]
- \(q\) Darcy velocity [m/s]
- \(\phi\) porosity
- \(h q_0\) heat flux across the interface between the aquifer and the adjacent layer [°C m/s]
- \(D\) thermal diffusivity of the aquifer [m²/s]
- \(v\) heat velocity [m/s]

The subscript \(f\) and \(s\) refer to the porous fluid and the porous solids, and the accent refers to the adjacent layer. Flow and conduction in the aquifer is along the \(x\)-direction. Bleeding (conduction) in the adjacent layer is in the \(z\)-direction. Flow in a porous medium induces dispersion due to scatter at smaller scale. Following the approach of Bear (2003), the thermal hydrodynamic macrodispersion is

\[
D = \frac{\lambda}{\rho c} + A_L \frac{\phi f_c f}{\rho c} q = \frac{\lambda}{\rho c} + A_L v
\]  
(2b)

consisting of the conductive part and the mechanical part. Here, \(A_L\) is the longitudinal mechanical macrodispersivity.

The conditions for the system are:

\[
D, D', h, v, A_L, H \quad \text{constant}
\]
(3a)

\[
T = T' = 0, x > 0, z > 0, t < 0
\]
(3b)

\[
T = T' = 0, \text{for } (x, z) \rightarrow \infty, t > 0
\]
(3c)

\[
T = T_i U[t], \text{at } x = 0 \text{ (jump at } t = 0) \quad \text{(3d)}
\]

\[
Q = 0, \quad \text{for } x > 0, t > 0
\]
(3e)

Here \(U[t]\) is the unit step function: \(U = 0\) for \(t < 0\) and \(U = 1\) for \(t > 0\).

3. Analytical solutions

Laplace transform is applied to equation (1), without the source term, and (3).

\[
D \frac{\partial^2 \Theta}{\partial x^2} - \frac{\partial \Theta}{\partial x} = \frac{s \Theta - D' \frac{\partial \Theta}{\partial z}}{H \frac{\partial \Theta}{\partial z}} \quad \text{in the domain } x > 0, s > 0
\]
(4a)

\[
D' \frac{\partial^2 \Theta'}{\partial z^2} = s h \Theta' \quad \text{in the domain } z > 0, s > 0
\]
(4b)

\[
\Theta = \Theta' = 0
\]
for \(x \rightarrow \infty\) and for \((z \rightarrow \infty, x > 0)\)
(5c)

\[
\Theta = \frac{T_i}{s}
\]
at \(x = 0\)
(5d)

With regard to condition (5c) the solution for equation (4b) is

\[
\Theta' = \Theta \exp\left[-s \frac{z}{\sqrt{sh}} \frac{H}{D'}\right] \Rightarrow
\]

\[
D' \frac{\partial^2 \Theta'}{\partial z^2} = -D' \frac{\sqrt{sh}}{D'} \Theta' = -D' \frac{\sqrt{sh}}{D'} \Theta
\]
and equation (4a) becomes

\[
D \frac{\partial^2 \Theta}{\partial x^2} - \frac{\partial \Theta}{\partial x} = \left(s + \frac{\sqrt{sh}}{H} \Theta\right)
\]
(6)

Trial by \(\Theta = B \exp[ax]\) provides the so-called characteristic equation

\[
D a^2 - va - \left(s + \frac{\sqrt{sh}}{H} \right) = 0
\]

It has one positive and one negative real root.
With regard to condition (5c) the negative root applies, and the solution of (6) becomes with condition (5d) and \( B = \theta = T_1/s \)

\[
\Theta = \frac{T_1}{s} \exp \left[ \frac{x}{2D} \right] \exp \left[ -x \sqrt{\frac{v}{2D}} + s + \sqrt{\frac{shD^*}{H^*}} \right]
\]  
(7)

The exact solution can be found by Laplace inverse transform (see Barends, 2009). In literature, Lauwerier’s solutions is mentioned for a special case of (7), which is discussed and extended here.

4. Convection, conduction and bleeding

Lauwerier assumes that heat transfers in the aquifer just by convection (no conduction) and into the adjacent layers by vertical conduction. So the heat distribution in the reservoir is assumed independent vertically uniform.

In this case, equation (7) can be used with \( D = 0 \). The elaboration for \( D \) tending to zero with the use of Taylor expansion can be simplified and by the inverse Laplace transform (Bateman, 1954a), the solution becomes:

\[
T = T_1 \text{erfc} \left[ \frac{s}{H} \sqrt{\frac{hD^*}{4v(\nu_t - x(1 - \delta))}} \right] U \left[ vt - x(1 - \delta) \right]
\]

where \( \delta = \frac{hDD^*}{\nu^2H^*} \)

(8)

The conditions related to the elaboration with Taylor expansion are:

\[
t > \frac{D}{v^2}, \quad t > \frac{4H^2}{hD^*} \quad \text{and} \quad t > \frac{4D}{v^2} \frac{hDD^*}{\nu^2H^*}
\]

Since the interest is focused on the behavior of the front at larger times, these conditions can be easily satisfied. The mathematical formulation is reformulated using following dimensionless variables:

\[
\chi = \frac{xhD^*}{\nu^2H^*}, \quad \tau = \frac{hD^*}{H^*}
\]

\[
T = T_1 \text{erfc} \left[ \frac{\chi}{2\sqrt{\tau - \chi(1 - \delta)}} \right] U \left[ \tau - \chi(1 - \delta) \right]
\]

(9)

The Lauwerie solution is similar to equation (9) when adopting \( \delta = 0 \). Formula (8) is reformulated for a specific reference temperature \( T_0 \) and injected temperature of \( T_1 \) according to:

\[
\frac{T - T_0}{T_1 - T_0} = \text{erfc} \left[ \frac{x}{H} \sqrt{\frac{hD^*}{4v(\nu_t - x(1 - \delta))}} \right] U \left[ vt - x(1 - \delta) \right]
\]

(10)

Figure 2. Isothermal lines in the \((\chi, \tau)\) plane for equation (9).

Figure 2 shows the effect of \( \delta \) in the extended Lauwerier solution, equation (9).

Next, the parameter effect consider using the equation (10). To be able to see the effects more clear each parameter has been changed 1, 2, 5 and 10 times. Some of these values may be are not realistic.

Figure 3 shows the sensitivity of the solution to fluid velocity. As expected, convection has a large effect on the breakthrough time.
Heat loss (bleeding) is more affective in thin aquifers, figure 4.

As the thermal conductivity of the top layer increase the bleeding effect becomes higher (figure 5). In the case of cold water injection to the hot aquifer, top layer cools off
while heating the aquifer which became colder due to injected water.

Graphs in figure 6 show the reservoir conductivity effect on heat transport. When convection (velocity) increases thermal diffusivity effects become less dominant, even vanish.

5. **Comparison with COMSOL Multiphysics**

The numerical calculations have been elaborated using Comsol Multiphysics 3.5. Here, in a hot aquifer with original temperature of $T_0 = 80$ degrees Celsius, confined between two impermeable layers, a hot water doublet system is installed, the filters placed at a distance of 200m. Cold water of $T_1 = 30^\circ C$ will be injected. The numerical simulation shows how and when cold water reaches the pumping well (breakthrough). The following assumptions were considered:

1. The aquifer is homogeneous and infinite in horizontal direction; its thickness $H$ is constant.
2. The caprock and the bedrock, above and below the aquifer, are homogeneous and impermeable.
3. The aquifer is located at the 2000 m depth.
4. Fluid flow in the aquifer is assumed to be steady; injection rate $Q$ is equal to the production rate.
5. Wells fully penetrate the aquifer.
6. Thermal equilibrium is supposed to take place instantaneously between the water and the rock matrix in the aquifer.
7. Volumetric heat capacity ($\rho c$) for both the water and the rock, and the caprock, are constant.
8. Differences in viscosity between injected water and initial water are disregarded.
9. Heat loss (bleeding) occurs only through top layer (caprock). The lower boundary is considered a thermal isolator.
10. The thermal and hydraulic properties of the ground are estimated from a realistic situation (appendix).
11. Aquifer thickness is kept small in order to better compare with the 1D analytical solution.
12. The distance between injection and pumping well is 200 m which is smaller than reality, for reasons of calculation time.
Figure 8 compares several analytical solutions for different conditions with numerical results. The blue line shows pure convection in the aquifer with a sharp front. The pink line demonstrates the solution of Ogata and Banks (1961) which deals with convection and conduction in the aquifer without bleeding. The pink line can perfectly illustrate the conduction effect when compared with the blue line. The red line is the extended Lauwerier solution. The green painted area between these lines shows the bleeding effect which is significant. Numerical result by COMSOL is the brown line.

Figure 9 and 10 show the numerical result compared to the extended Lauwerier solution. The difference is painted in pink. It is due to numerical error (numerical dispersion) and a mismatch of the semi-2D analytical solution and the full 2D numerical solution.

Numerical errors in models can lead to mis-evaluation of the geothermal reservoir life time and heat potential which is of economic importance.

6. Conclusion

The extended Lauwerier solution which considers both conduction-convection in the aquifer and conduction in the adjacent layer (bleeding) shows that conduction in the aquifer is significant especially in the case of low flow velocity. A comparison which was made between a numerical calculation by COMSOL and the extended Lauwerier solution shows a serious deviation which is mainly caused by numerical dispersion.

There are methods to decrease numerical errors but in general errors are inevitable. An analytical solution can provide an idea about the size of numerical errors. Understanding of numerical errors can provide more accurate interpretation of numerical simulations.

7. References


8. Appendix

Table 1: The parameters used for numerical calculations

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>0.25 (porosity)</td>
</tr>
<tr>
<td>$k$</td>
<td>2.8E-13[m$^2$] (permeability)</td>
</tr>
<tr>
<td>$\rho_f$</td>
<td>1200[kg/m$^3$] (Saline warm fluid density)</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>2800[kg/m$^3$] (solid density)</td>
</tr>
<tr>
<td>$\lambda_s$</td>
<td>1.7[W/(m.degC)] (Heat conductivity of rock)</td>
</tr>
<tr>
<td>$\lambda_f$</td>
<td>0.6[W/(m.degC)] (Heat conductivity of fluid)</td>
</tr>
<tr>
<td>$H$</td>
<td>15[m] (thickness)</td>
</tr>
<tr>
<td>$T_c$</td>
<td>30[degC] (cold temperature)</td>
</tr>
<tr>
<td>$T_f$</td>
<td>80[degC] (warm temperature)</td>
</tr>
<tr>
<td>$c_f$</td>
<td>4184[J/kg/degC] (specific heat capacity of fluid)</td>
</tr>
<tr>
<td>$c_s$</td>
<td>830[J/kg/degC] (specific heat capacity of solid)</td>
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<tr>
<td>$k_t$</td>
<td>1E-18[m$^2$] (permeability of top layer)</td>
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<tr>
<td>$\phi_t$</td>
<td>0.05 (porosity of top layer)</td>
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<tr>
<td>$\rho_t$</td>
<td>3000[kg/m$^3$] (density of top layer)</td>
</tr>
<tr>
<td>$\lambda_t$</td>
<td>2.6[W/(m.degC)] (thermal conductivity of top layer)</td>
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<td>840[J/kg/degC] (heat capacity of top layer)</td>
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<tr>
<td>$T_t$</td>
<td>80[degC] (top layer temperature)</td>
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