Solving the Paraxial Wave Equation using COMSOL

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Present focus

- Free-space propagation of a Gaussian-beam wave as described by the paraxial wave equation.
- Comparison of analytic solutions to those obtained numerically using COMSOL.

Longer-term interests/goals

- Non-uniform medium, non-linear effects*.
- Part of a broader directed-energy research initiative at USNA (engineering, mathematics, physics).
- **Involving midshipmen in research.**

*Mark J. Schmitt, "Mitigation of thermal blooming and diffraction effects with high-power laser beams", J. Opt. Soc. Am. B 20, 719-724 (2003) .

The paraxial wave equation

Larry C. Andrews and Ronald L. Phillips
Laser Beam Propagation through Random Media, 2nd ed.

- Start with the standard wave equation.
- Build in beam propagation along the z-axis.

$$\begin{cases} \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \right) u = \dfrac{1}{c^2} \dfrac{\partial^2 u}{\partial t^2} \\ u(x, y, z, t) = V(x, y, z) \mathrm{e}^{i(kz - \omega t)} \end{cases}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) V + \frac{\partial^2 V}{\partial z^2} + 2ik \frac{\partial V}{\partial z} = 0$$

The paraxial wave equation

- (transverse spreading) << (propagation distance).
- V is complex.

$$\begin{cases} \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)V+\cancel{\frac{\partial^2 V}{\partial z^2}}+2ik\frac{\partial V}{\partial z}=0 \\ V=V_1+iV_2 \end{cases}$$

$$\begin{cases} \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)V_1-2k\frac{\partial V_2}{\partial z}=0 \\ \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)V_2+2k\frac{\partial V_1}{\partial z}=0 \end{cases}$$

Lowest-order Gaussian-beam wave

some definitions,

$$r = \sqrt{x^2 + y^2}\ , \quad \Theta_0 = 1 - \frac{z}{F_0}\ , \quad \Lambda_0 = \frac{2z}{k W_0^2}\ .$$

longitudinal phase shift: $\phi(z) = \tan^{-1}\frac{\Lambda_0}{\Theta_0}$

spot size radius: $W(z) = W_0\sqrt{\Theta_0^2 + \Lambda_0^2}$

radius of curvature: $F(z) = \dfrac{F_0(\Theta_0^2 + \Lambda_0^2)(\Theta_0 - 1)}{\Theta_0^2 + \Lambda_0^2 - \Theta_0}$

Lowest-order Gaussian-beam wave

$$\begin{cases} V_1(r,z) = +\dfrac{W_0}{W}\exp\left(-\dfrac{r^2}{W^2}\right)\cos\left(\phi+\dfrac{kr^2}{2F}\right) \\ V_2(r,z) = -\dfrac{W_0}{W}\exp\left(-\dfrac{r^2}{W^2}\right)\sin\left(\phi+\dfrac{kr^2}{2F}\right) \end{cases}$$

$$I^0(r,z) = V_1^2 + V_2^2 = \frac{W_0^2}{W^2}\exp\left(-\frac{2r^2}{W^2}\right)$$

$$W(z) = \frac{W_0}{\sqrt{I^0(0,z)}} = \frac{W_0}{\sqrt{V_1(0,z)^2 + V_2(0,z)^2}}$$

Lowest-order Gaussian-beam wave

At the location of the *beam waist*,

- The spot size is a minimum.
- The intensity is a maximum.
- The beam is collimated, $F/F_0 = \pm\infty$.
- Transition from converging to diverging.

$$\Omega_f = \frac{2F_0}{kW_0^2}\,, \quad z_B = \frac{F_0}{1+\Omega_f^2}\,, \quad W_B = W_0\sqrt{\frac{\Omega_f^2}{1+\Omega_f^2}}$$

Punch Line: COMSOL has difficulty passing through the beam waist / collimation.

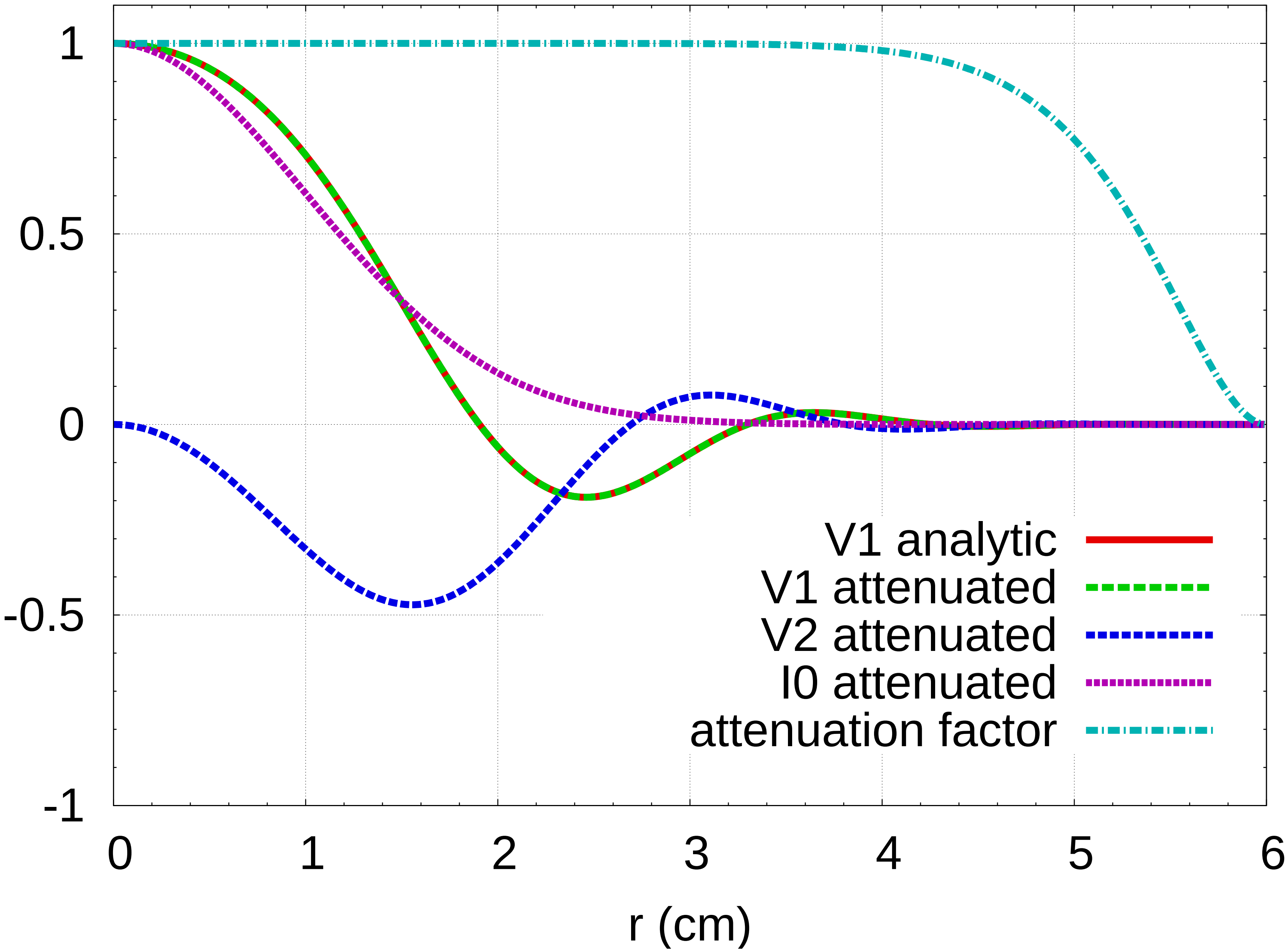
Setting up the COMSOL run

$$\begin{cases} k = 2\pi/\lambda \,, \quad \lambda = 532\,\text{nm} \\ W_0 = 2\,\text{cm} \\ W_B = 1\,\text{cm} \end{cases}$$

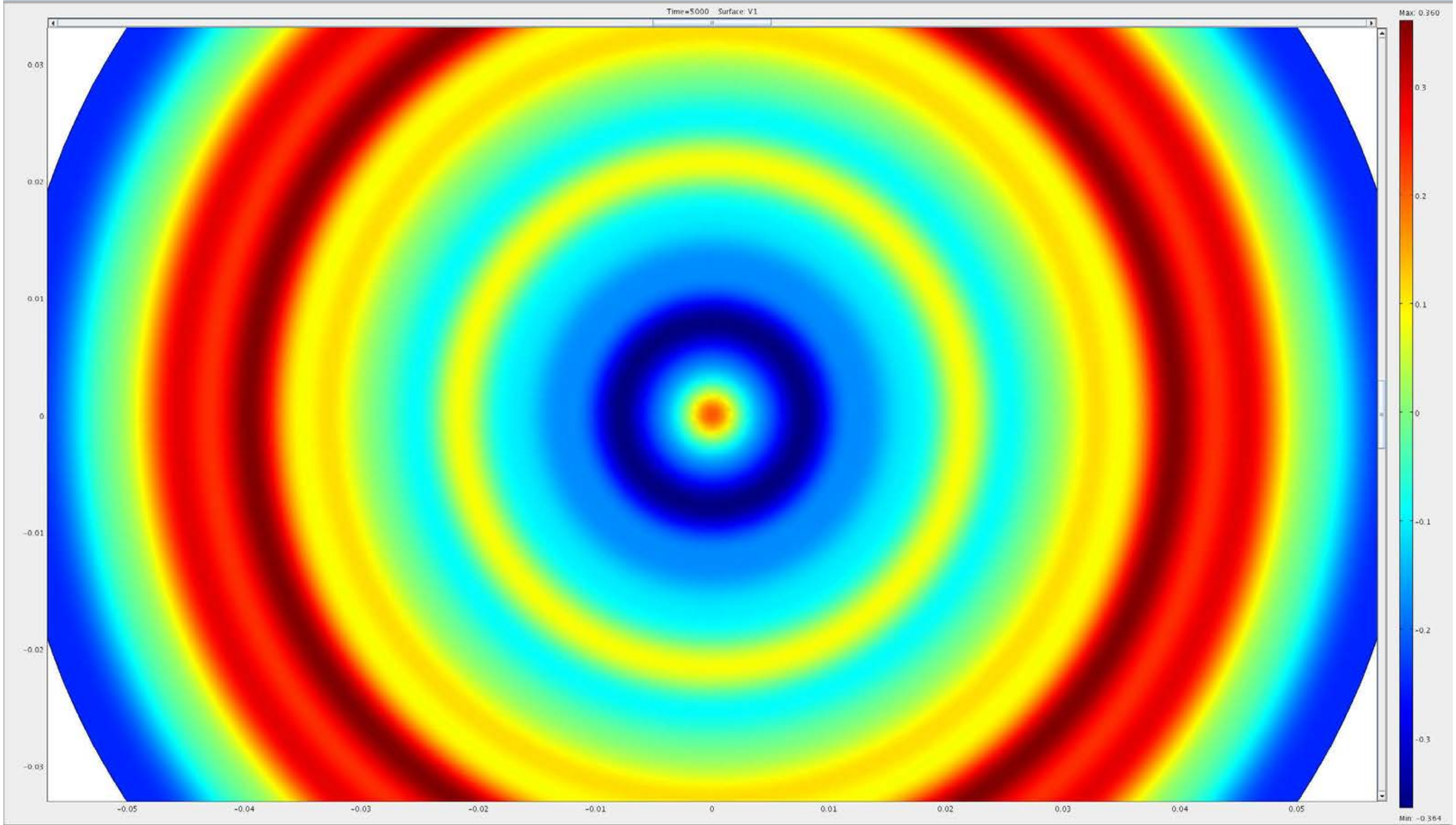
$$\begin{cases} F_0 = +1363.76\,\text{m} \\ z_B = 1022.82\,\text{m} \end{cases}$$

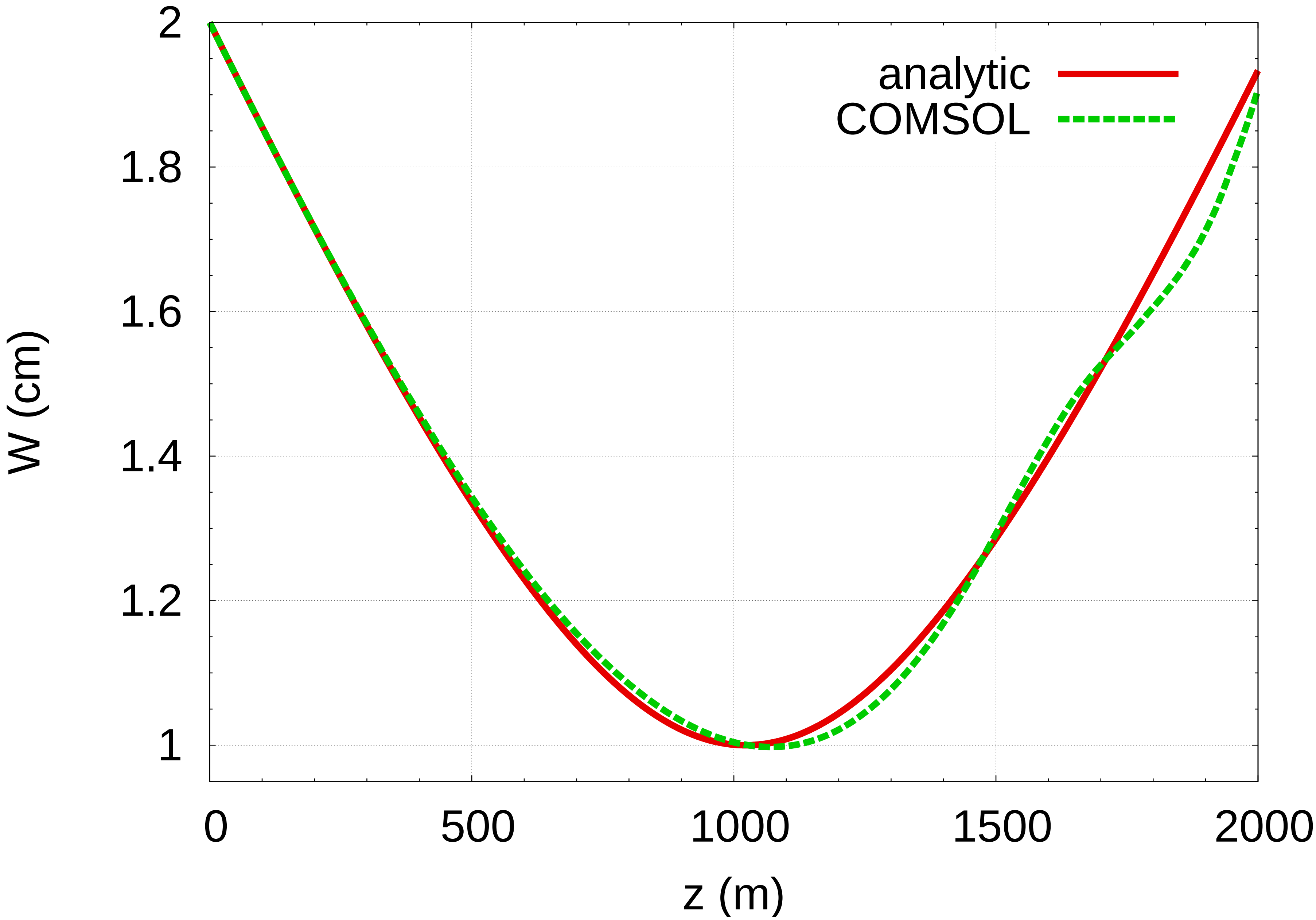
How best to implement a bounded domain in *xy*?

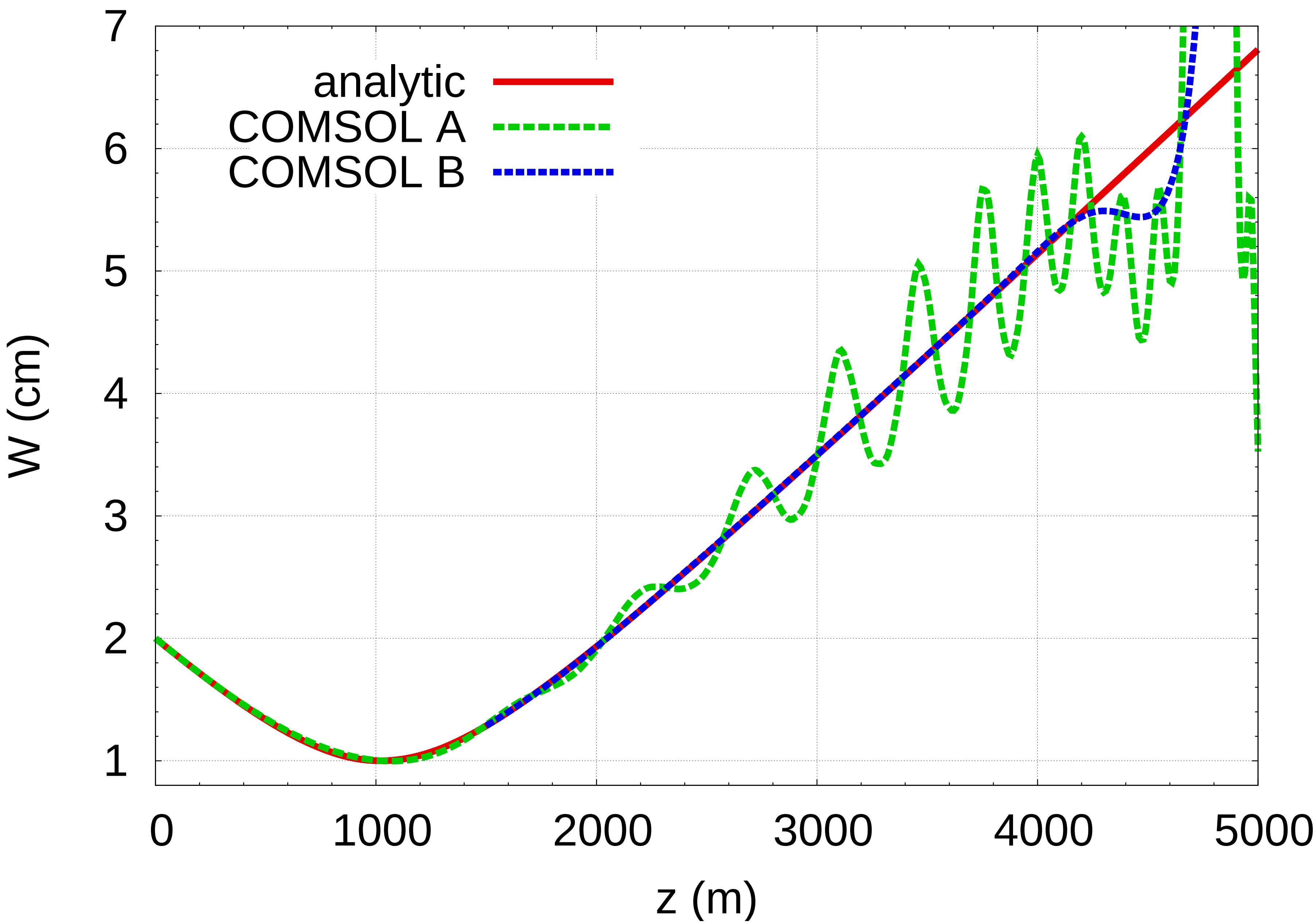
- Suitable for *z* range(0, 10 m, 2000 m).
- $R_{\text{boundary}} = 6\,\text{cm}$.
- $f_{\text{attenuation}} = 0.5\left(1 + \cos\left[\left(r/R_{\text{boundary}}\right)^6 \pi\right]\right)$.
- Neumann boundary condition.
- Meshing: extremely fine, refine mesh, 99432 elements.


1
0.5
0
-0.5
-1
0
1
2
3
4
5
6
r (cm)
V1 analytic
V1 attenuated
V2 attenuated
I0 attenuated
attenuation factor

V_1 at z = 5000 m, cylindrical symmetry maintained.




analytic
COMSOL
W (cm)
z (m)
2
1.8
1.6
1.4
1.2
1
0
500
1000
1500
2000



analytic
COMSOL A
COMSOL B
W (cm)
z (m)
7
6
5
4
3
2
1
0
1000
2000
3000
4000
5000